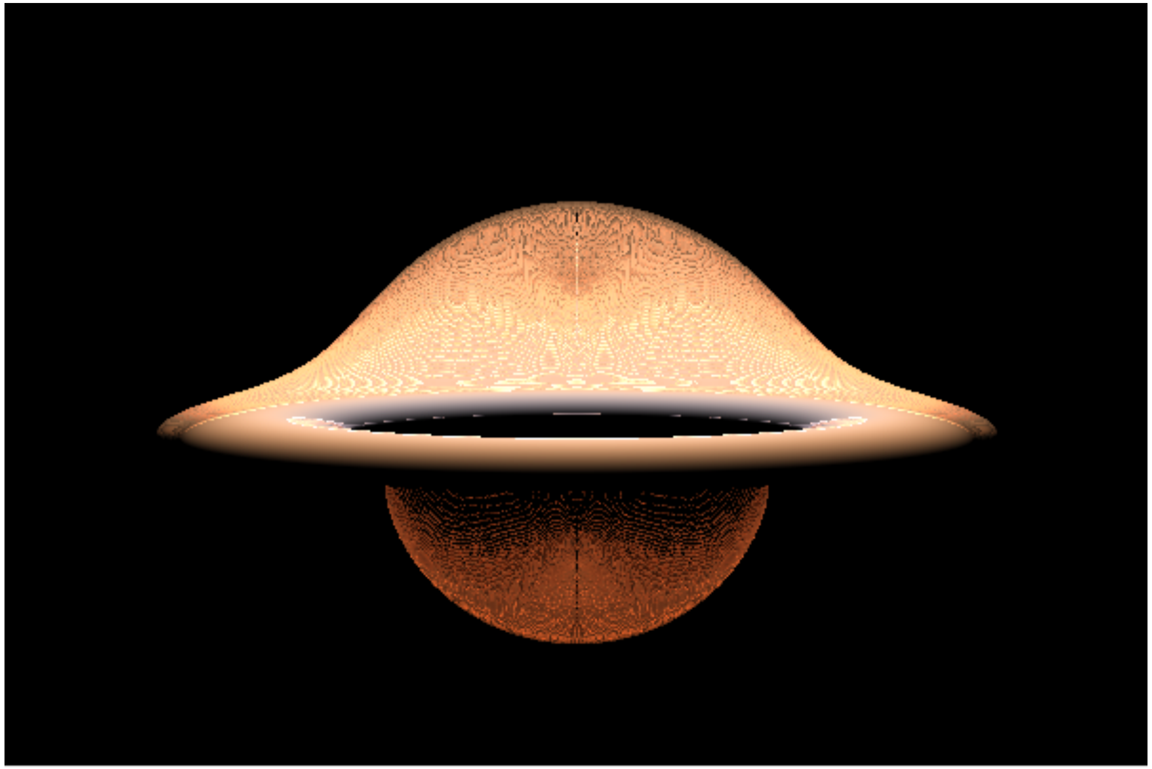
Geodesic lines in Schwarzschild und Kerr Metric

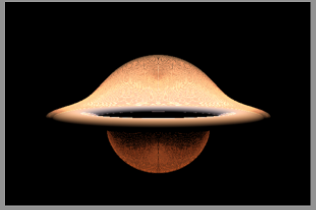
(Bachelor Thesis)



Overview

- problem
- thematic context
- problem (detailed)
- implementation
- programs
- results
- outlook

Problem



- How light is deflected close to a black hole?
- Effect on how an accretion disk looks like?

Thematic Context

- Newtonian mechanics:

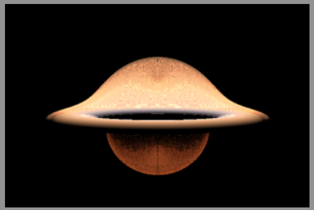
  mass distribution causes a gravitational field

- General theory of relativity:

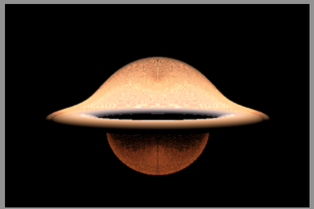
  mass distribution curves spacetime

  spacetime curvature is described by metric

  particles move along geodesic lines



Particle movement along geodesic lines



- distance between spacetime events becomes extremal (minimal)
- definition of a geodesic: $L_{BA} = \int_B^A ds$
- variation δ of the event x^μ :

$$\delta L_{BA} = -\frac{1}{c} \int_{B/c}^{A/c} \left[\frac{d^2 x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right] g_{\xi\sigma} \delta x^\xi \, d\tau$$

➔ for L_{BA} becoming extremal, the variation has to vanish

➔ the geodesic equation has to be fulfilled:

$$\frac{d^2 x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} = 0$$

Description of spacetime curvature by metrics

- Schwarzschild metric
  parameterized by mass *M*

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{1-r_s/r} + r^2\left(d\vartheta^2 + \sin\vartheta d\varphi^2\right)$$

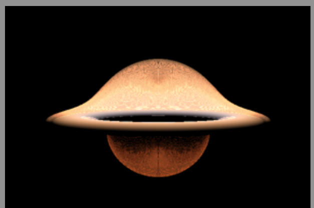
where $r_s = 2GM/c^2$ is the Schwarzschild radius

- Kerr metric
  parameterized by mass *M* and angular momentum *a*

$$ds^2 = -\rho^2\frac{\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2}{\rho^2}\left(d\varphi - \frac{2aMr}{\Sigma^2}dt\right)^2\sin^2\vartheta + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\varphi)^2$$

and the abbreviations

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\vartheta \\ \Delta &= r^2 - 2Mr + a^2 \\ \Sigma^2 &= (r^2+a^2)^2 - a^2\Delta\sin^2\vartheta\end{aligned}$$



➔ Black Holes occur from stellar collaps,
so angular momentum has to be taken into account

Accretion disk

- Standard accretion model:

  geometrically thin disk

  axial symmetry

  non-selfgravitating

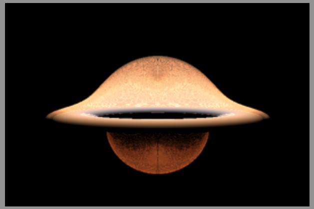
  optically thick

- viscose shear → heat in the disk
  → radial dependent effective temperature

$$T_{eff} = \left[\frac{3GM_* \dot{M}}{8\pi R^3} \left(1 - \frac{R_*}{R} \right)^{1/2} \right]^{1/4} = T_*(R/R_*)^{-3/4} \propto R^{-3/4}$$

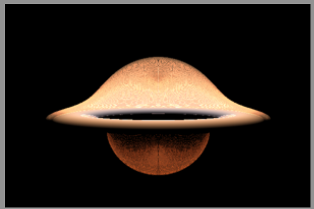
with $T_* = (\frac{3GM_* \dot{M}}{8\pi\sigma R_*^3})^{1/4}$.

→ generation of a spectrum



Problem (detailed)

- How light is deflected close to a Black Hole?
- Effect on how an accretion disk looks like?



Problem (detailed)

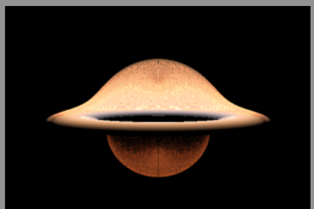
- How light is deflected close to a Black Hole?
- Effect on how an accretion disk looks like?

➔ integration of the geodesic equation

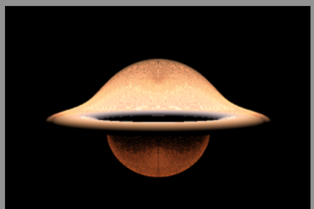
➔ development of a program for visualization of geodesic lines

➔ development of a raytracer

➔ calculation of the disk spectrum



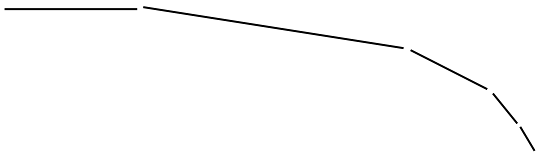
Implementation



- integration of the geodesic equation

  Runge-Kutta-Fehlberg (4,5) method for the integration of light-like geodesics

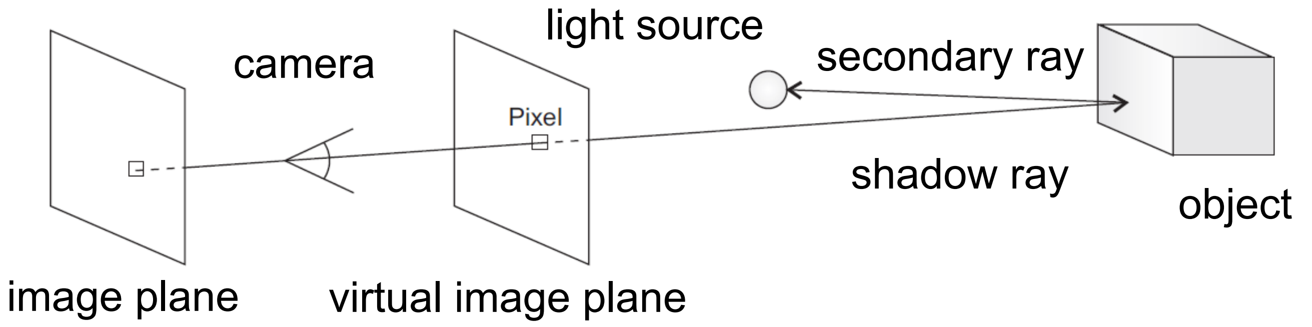
➔ geodesics can be approximated by multiple Euklidean line segments

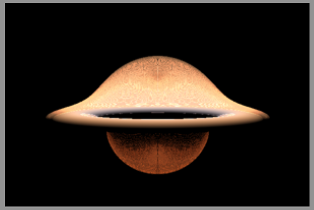


Implementation

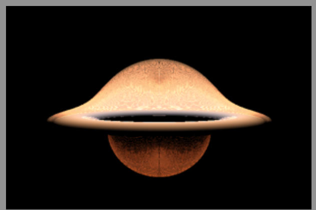
- Raytracing

at first Euklidean:





Implementation



- Raytracing

4 D (general relativity):

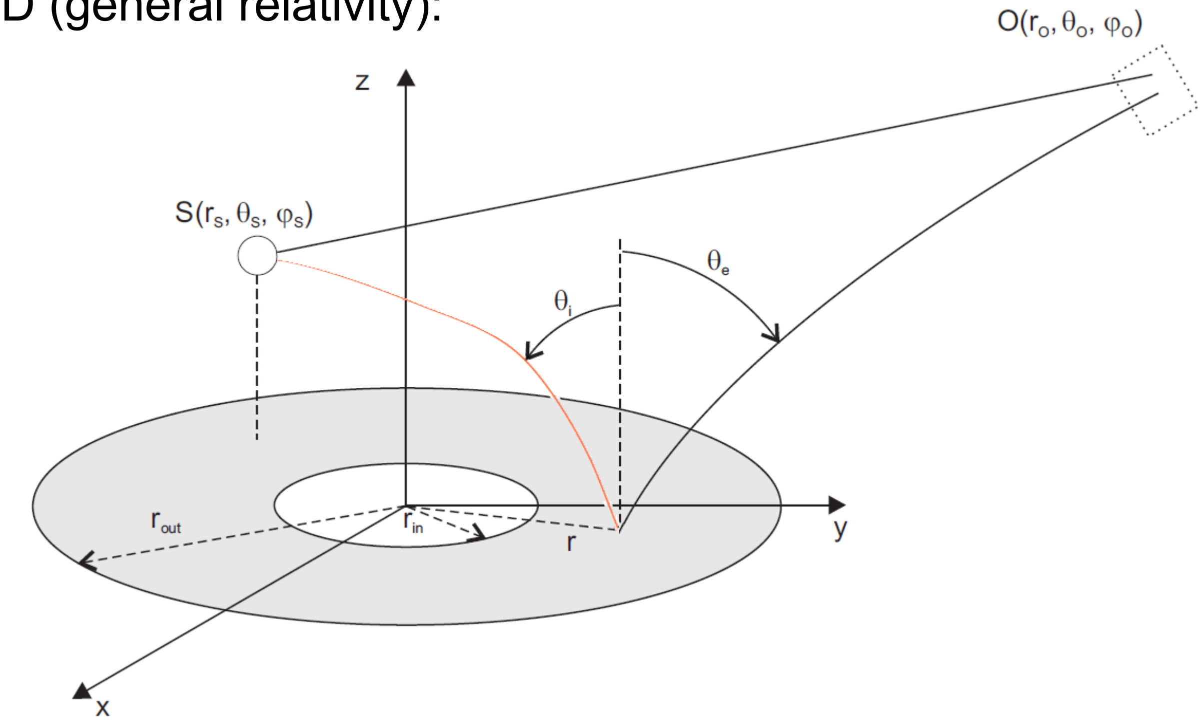
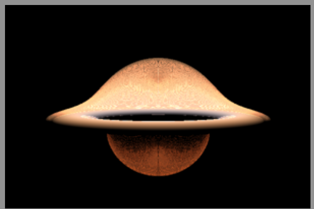


Diagram of relativistic raytracing:
Light comes from primary source ***S***, is reflected at the surface of the accretion disk and is re-emitted to observer ***O***

Implementation



- calculation of the disk spectrum

viscose shear → heat in the disk
→ radial dependent effective temperature

$$T_{eff} = \left[\frac{3GM_* \dot{M}}{8\pi R^3} \left(1 - \frac{R_*}{R} \right)^{1/2} \right]^{1/4} = T_* (R/R_*)^{-3/4} \propto R^{-3/4}$$

with $T_* = (\frac{3GM_* \dot{M}}{8\pi\sigma R_*^3})^{1/4}$.

$$\rightarrow \quad F_E = K \frac{4\pi E^3}{h^3 c^2} \int_1^\infty \frac{r}{\exp\left[E/kT_s(r)\right] - 1} dr$$ with $K = (\frac{R_*}{D})^2 cos(i)$

→ spectrum results from integration over the whole disk

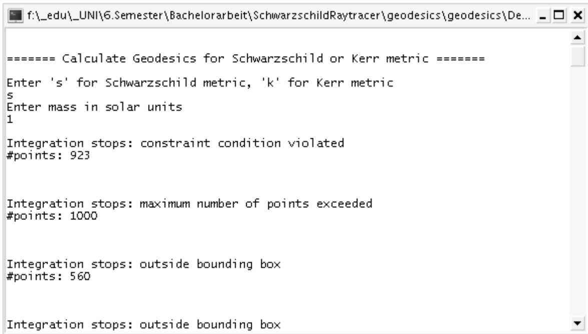
"multitemperature blackbody spectrum"

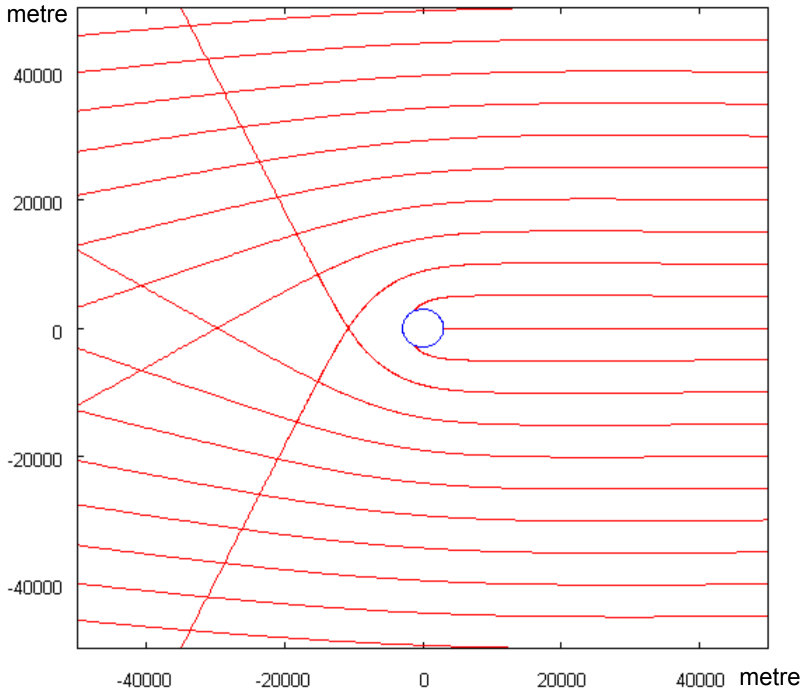
Programs

- program **`geodesics`**

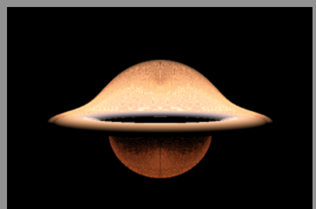
integration of geodesics via Runge-Kutta-Fehlberg (4,5) algorithm

visualization of geodesic lines with **`gnuplot`**





geodesic lines in Schwarzschild metric, M=1M*



Programs

- program **`accretion`**

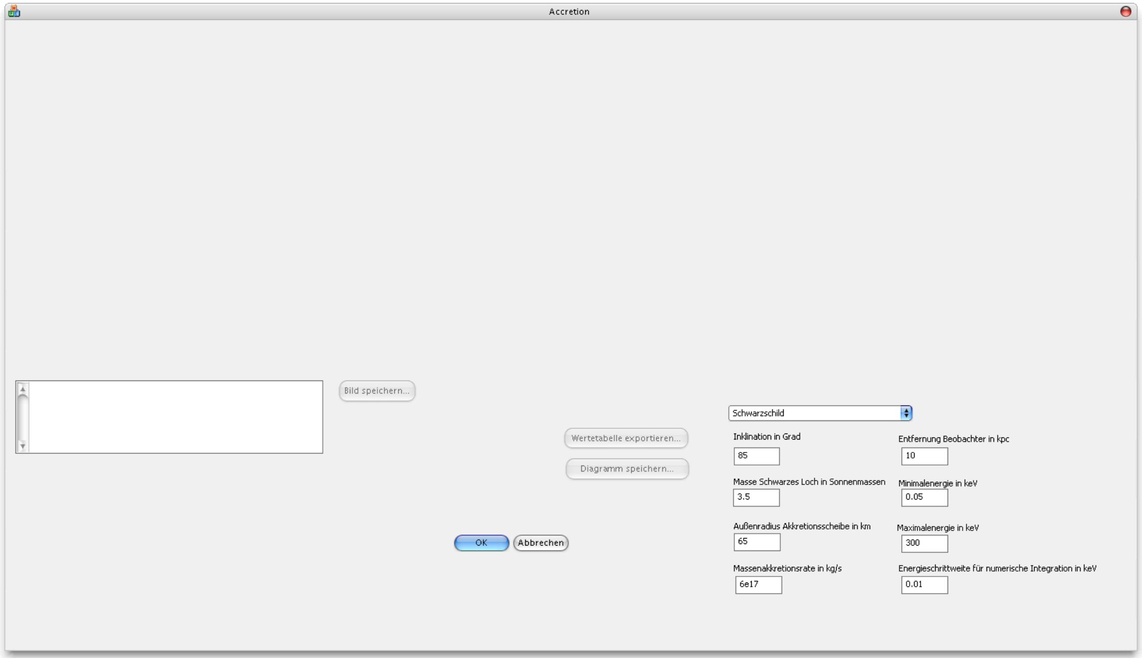
  integration of geodesics via Runge-Kutta-Fehlberg (4,5) algorithm

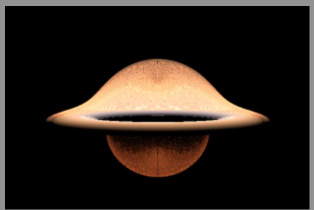
  raytracing of the accretion disk ‘s image under an inclination angle

  “multitemperature blackbody spectrum“

  visualization of the flux spectrum

  parallelization of the source code





Programs

- program **`accretion`**

  integration of geodesics via Runge-Kutta-Fehlberg (4,5) algorithm

  “multitemperature blackbody spectrum“

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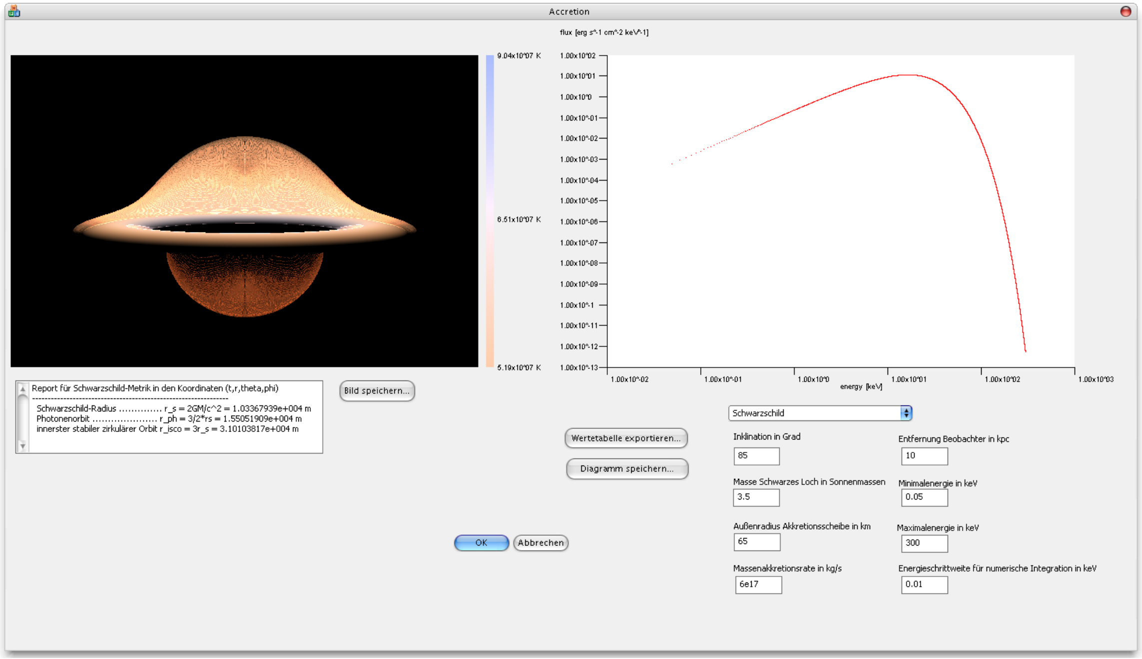
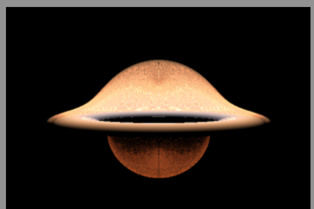
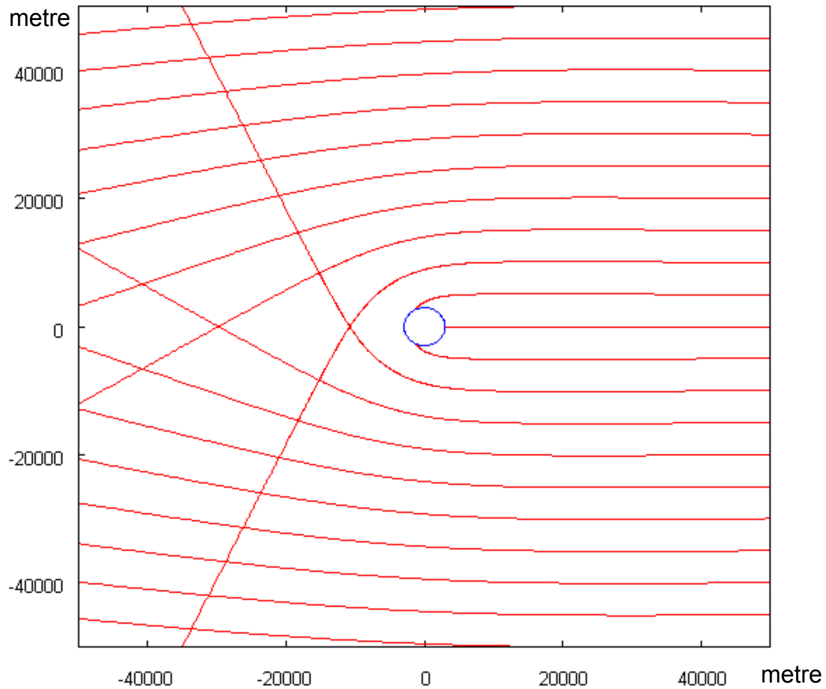


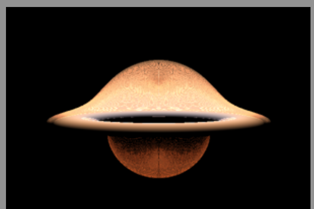


Results

- program **`geodesics`**

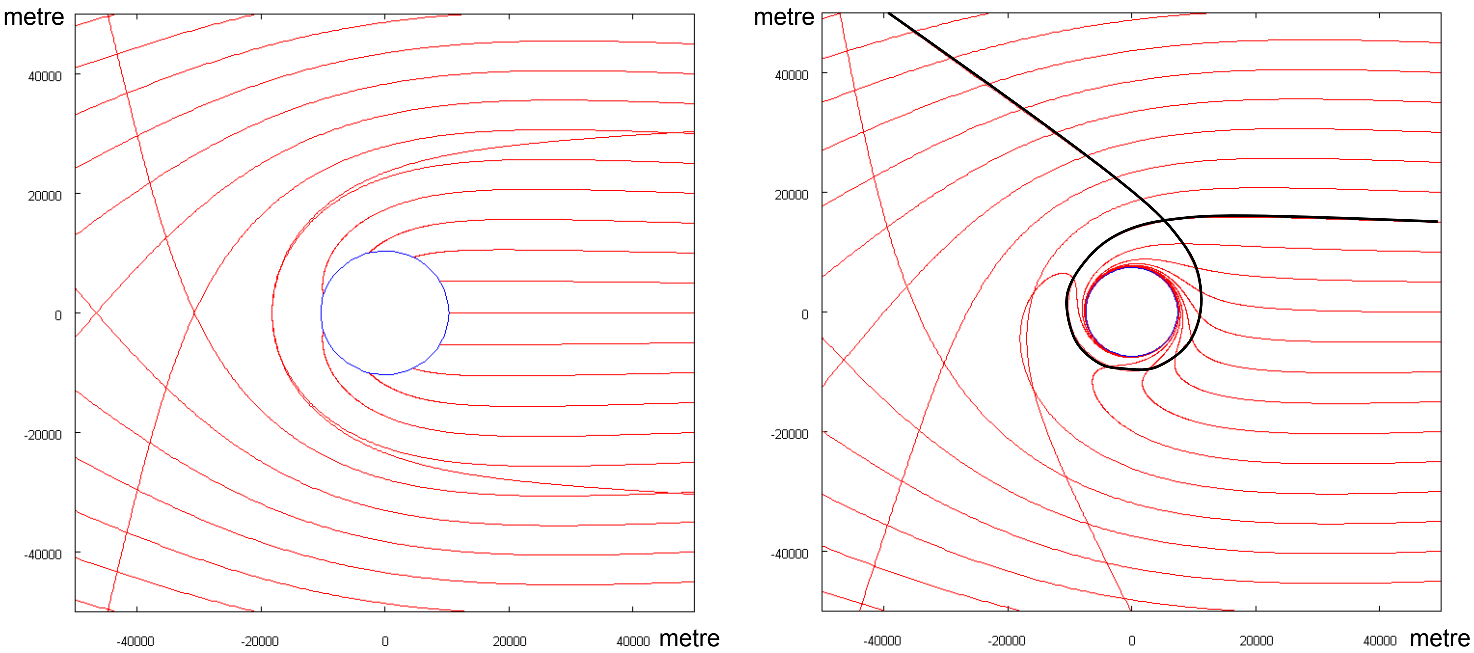


geodesic lines in Schwarzschild metric, $M=1M_*$

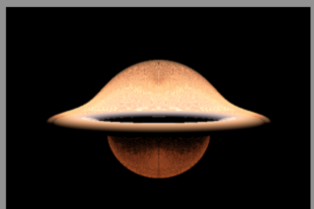


Results

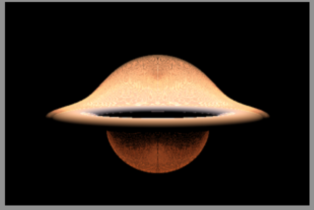
- program **`geodesics`**



geodesic lines in Schwarzschild and Kerr metric,
M=3.5 M_*, a/M=0.9

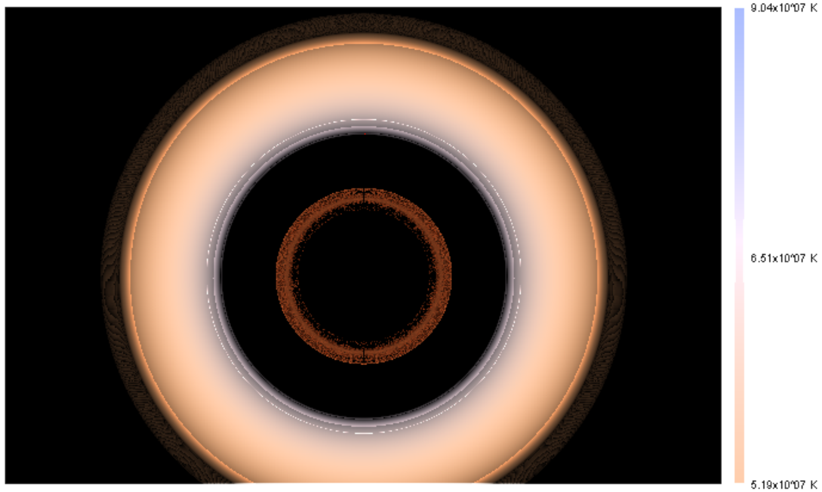


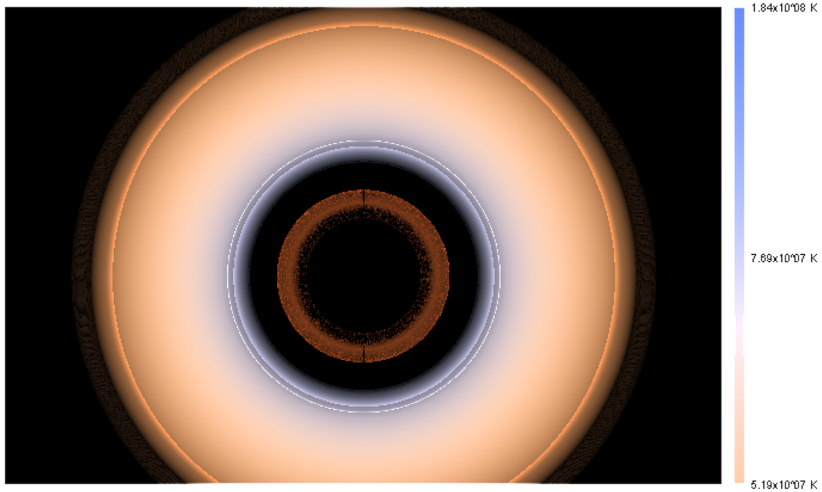
Results



- program **`accretion`**

  changing of the accretion disk ‘s image under an inclination angle





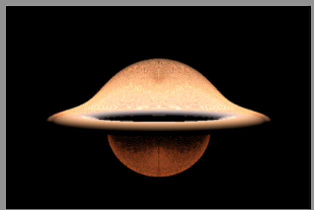
Schwarzschild and Kerr metric,
M=3.5 M*, a/M=0.9

outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

accretion rate: 6×10^{17} kg/s

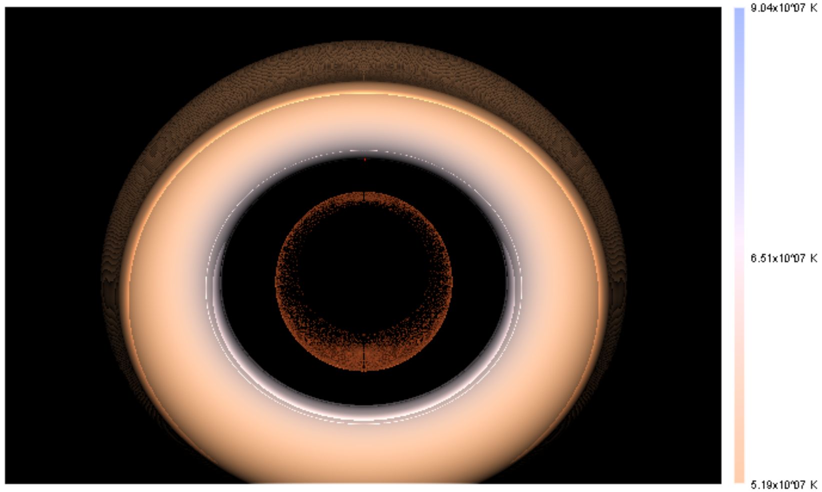
Inclination angle: 1 °

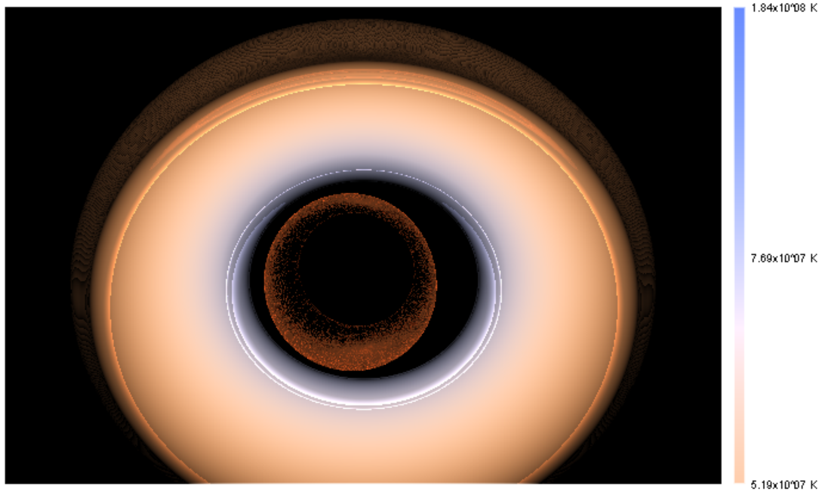
Results



- program **`accretion`**

  changing of the accretion disk ‘s image under an inclination angle





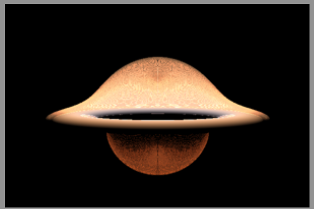
Schwarzschild and Kerr metric,
M=3.5 M*, a/M=0.9

outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

accretion rate: 6×10^{17} kg/s

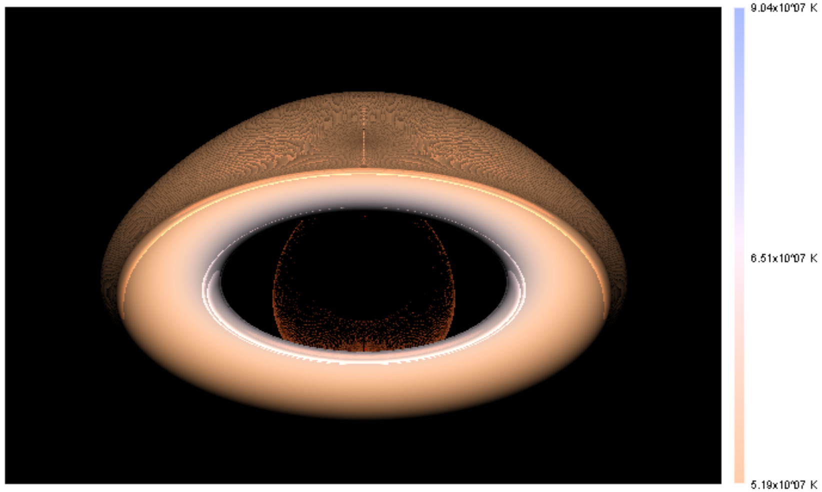
Inclination angle: 30 °

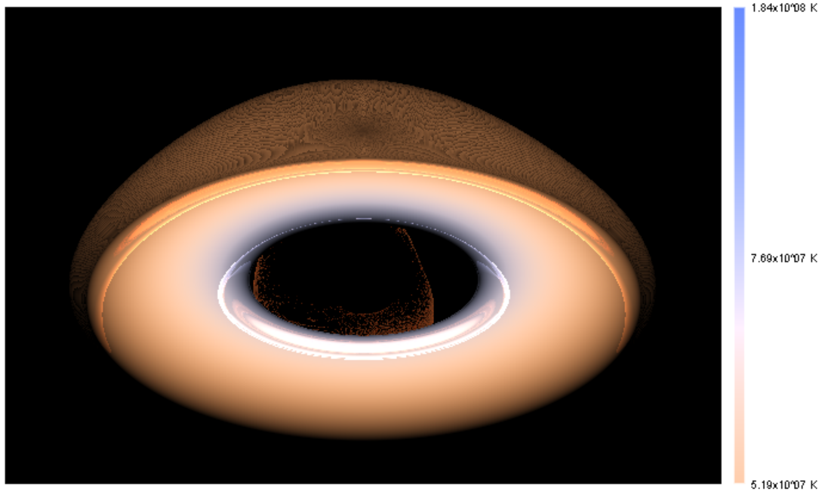
Results



- program **`accretion`**

  changing of the accretion disk ‘s image under an inclination angle





Schwarzschild and Kerr metric,
M=3.5 M*, a/M=0.9

outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

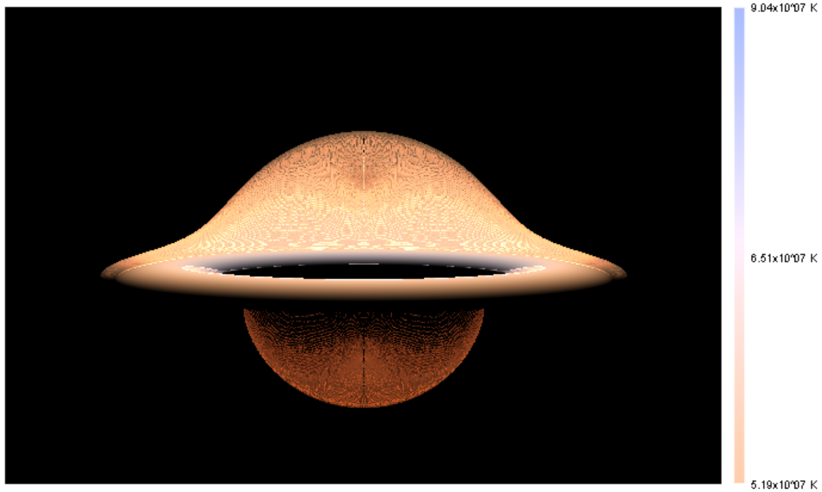
accretion rate: 6×10^{17} kg/s

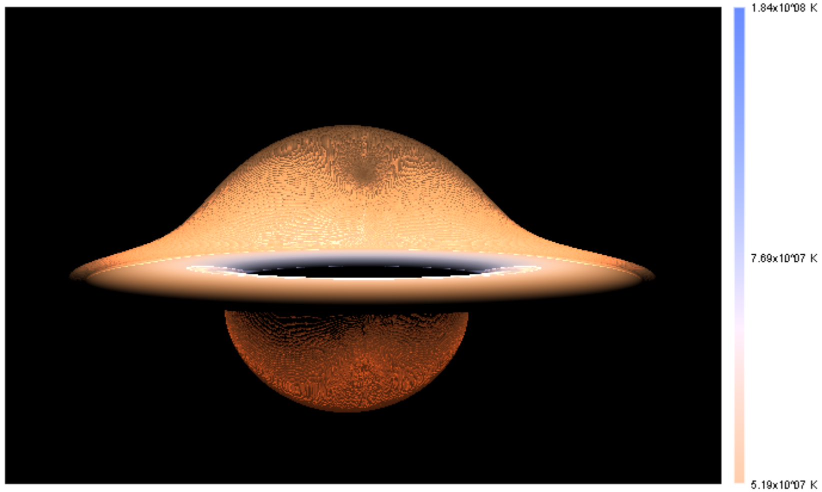
Inclination angle: 60 °

Results

- program **`accretion`**

  changing of the accretion disk ‘s image under an inclination angle



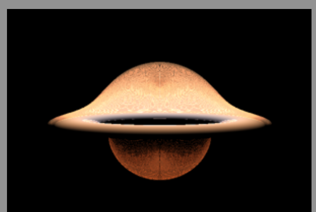


Schwarzschild and Kerr metric,
M=3.5 M*, a/M=0.9

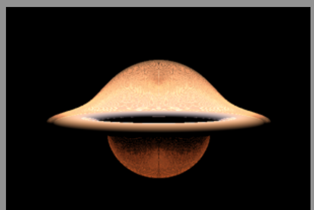
outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

accretion rate: 6×10^{17} kg/s

Inclination angle: 85 °

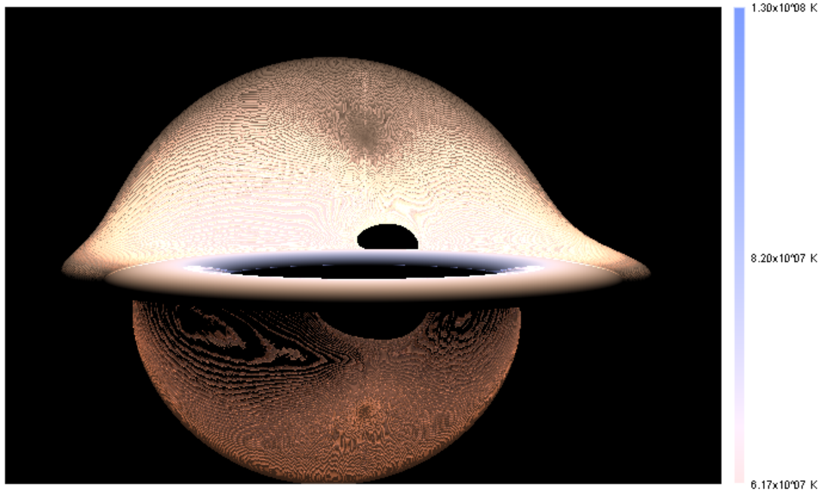


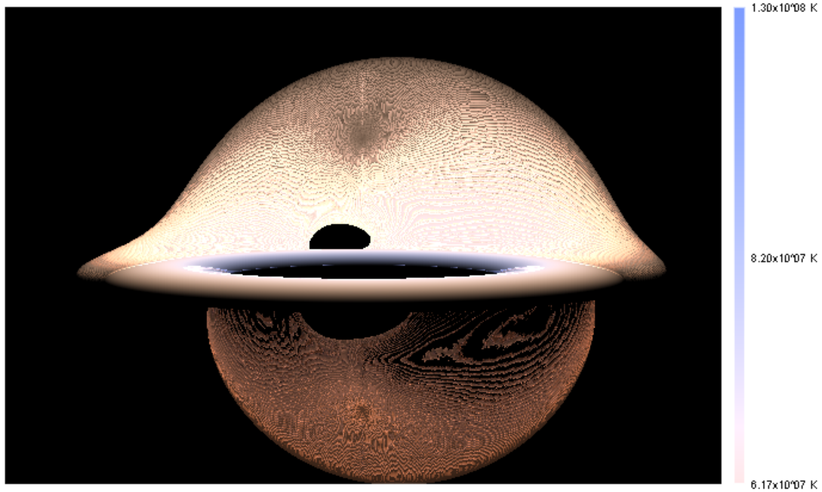
Results



- program **`accretion`**

  Kerr, retrograde and prograde





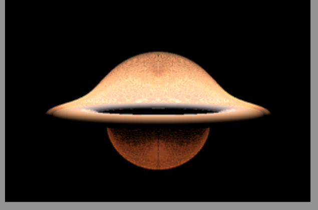
Kerr metric,
$M=7\ M_*$,
$a/M=0.9$ and $a/M=-0.9$

outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

accretion rate: 6×10^{17} kg/s

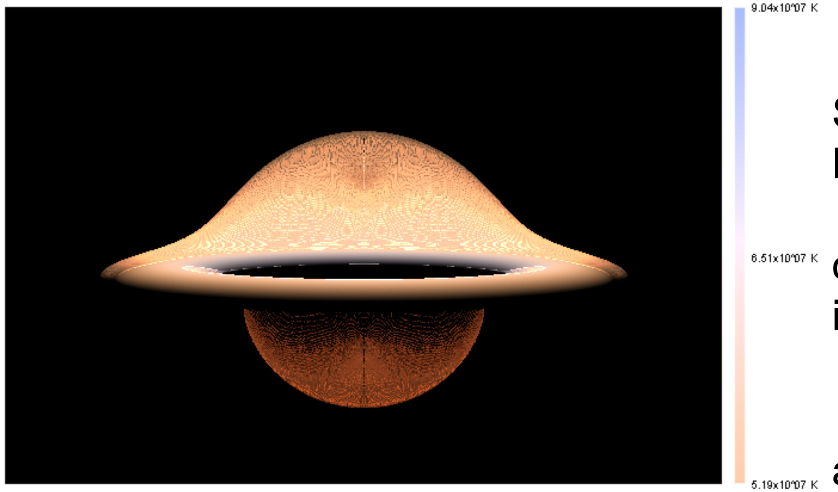
Inclination angle: 85 °

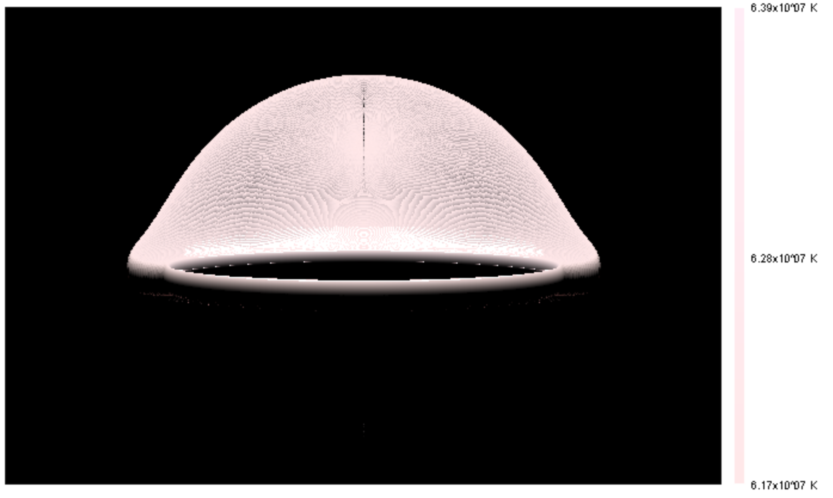
Results



- program **`accretion`**

  Schwarzschild, effect from central mass





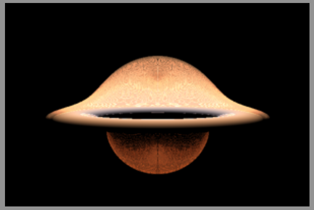
Schwarzschild metric,
M=3.5 M_* and M=7 M_*

outer disk radius: 6500 m
inner disk radius: innermost circular stable orbit

accretion rate: 6×10^{17} kg/s

Inclination angle: 85 °

Results



- program **`accretion`**

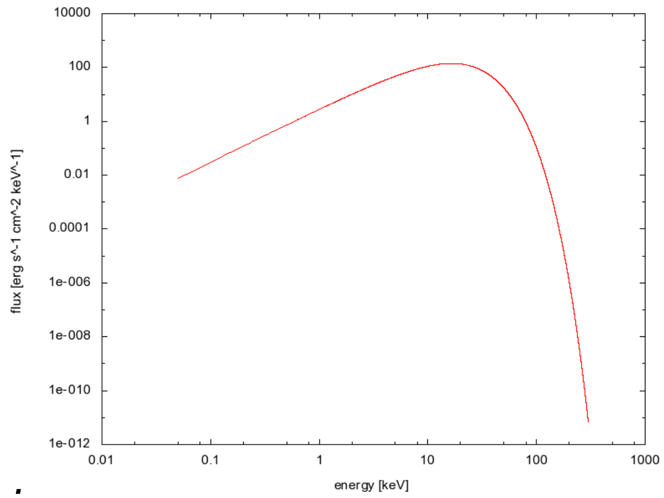
  spectral flux
  M=3.5 M*
  outer disk radius: 6500 m
  inner disk radius: innermost circular stable orbit
  inclination: 1 °, 85 °

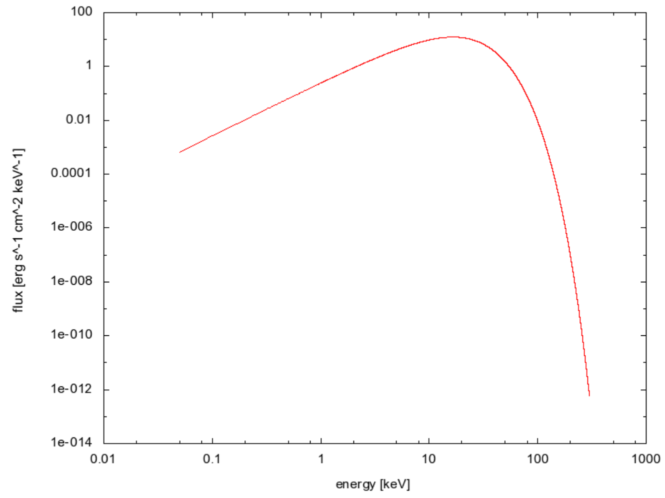
  → observed spectrum depends on cosine of inclinatiation angle *i*

  comes from

$$F_E = K\frac{4\pi E^3}{h^3c^2}\int_1^{\infty}\frac{r}{\exp\left[E/kT_s(r)\right]-1}\,dr$$

$$K = \left(\frac{R_*}{D}\right)^2 cos(i)$$





Results

- program **`accretion`**

  spectral flux
  M=3.5 M*
  outer disk radius: 6500 m
  inner disk radius: innermost circular stable orbit
  inclination: 1 °, 85 °

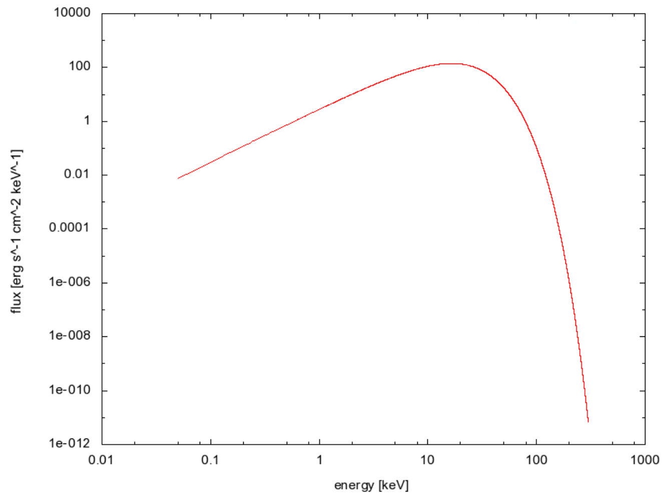
  → range of linearity in double logarithmic plot

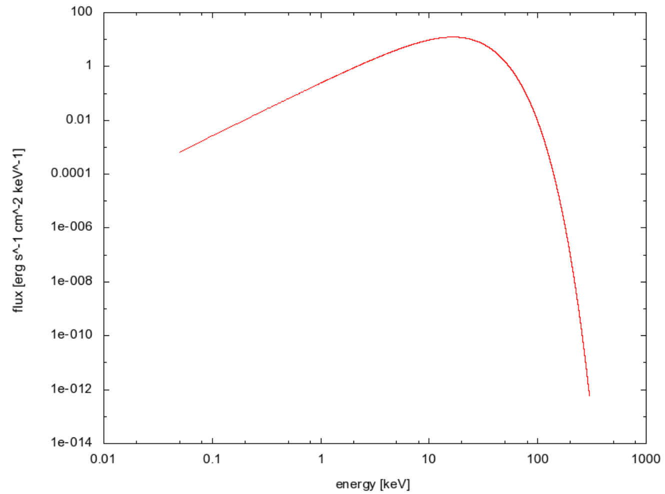
  comes from

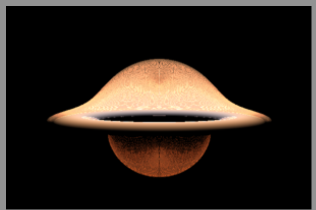
$$T_{eff} = \left[\frac{3GM_*\dot{M}}{8\pi R^3}\left(1-\frac{R_*}{R}\right)^{1/2}\right]^{1/4}$$

$$= T_*(R/R_*)^{-3/4} \propto R^{-3/4}$$

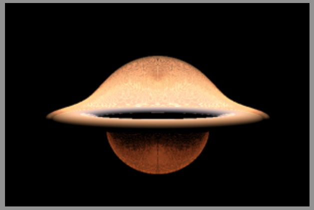
$$F_\nu \propto \nu^{3-(2/(1/3))} = \nu^{1/3}$$





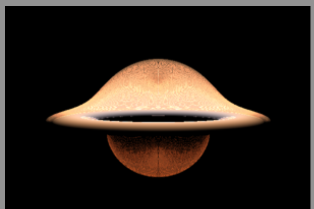


Results - Summary



- deflection of light: clearly to see at inclination angle of 85 °
- upper side of accretion disk can fully be seen, lower side can partially be seen
- asymmetry in Kerr metric (prograde, retrograde)
- light rays of the second kind (resulting in circular structure around event horizon)
- temperature of accretion disk depends on central mass and accretion rate

Results – Technical aspects



- computation time

  developed with C++ under Visual Studio 2008

  parallelized with OpenMP

  running at AMD Athlon X2 5050e, Dual Core, 2 x 2.6 GHz

M in M_*	*a/M*	computation time without parallelisation	computation time with parallelisation	factor without/with parallelisation
0.1	0	3:17	2:15	1.46
3.5	0	3:30	2:38	1.33
3.5	0.9	3:44	2:55	1.28
7.0	0.9	5:02	3:00	1.68

→ computation time depends on number of line segments (adaptive Runge-Kutta-Fehlberg (4,5) algorithm

→ with OpenMP up to 1.68 times faster on this machine for given tests

Outlook

- implementation of further physical effects

  gravitational red shift

  Doppler red shift/ blue shift

- implementation of more complex accretion disk model

  Inhomogenities in density and thickness

- implementation of other metrics

  e.g. Reissner-Nordström metric, Kerr-Newman metric